Plato, Brouwer, and classification

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In a nutshell

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*The typical constructivist response to a nonconstructive mathematical theorem is to modify the theorem by adding hypotheses or “extra data”. In contrast, our approach in this book is to analyze the provability of mathematical theorems as they stand, passing to stronger subsystems of $\mathbb{Z}_2$ if necessary.* (SOSOA, p. 32)
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The final sentence is somewhat paradoxical as follows.
Coding ordinary mathematics

All here shall known $\varepsilon$-$\delta$-continuity for $f : [0, 1] \rightarrow \mathbb{R}$ as follows:

$$(\forall \varepsilon > 0, x \in [0, 1])(\exists \delta > 0)(\forall y \in [0, 1])(|x - y| < \delta \rightarrow |f(x) - f(y)| < \varepsilon).$$
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Now compare this to ‘continuity-via-codes’ in $L_2$ from SOSOA:

**II.6. Continuous Functions**

**Definition II.6.1** (continuous functions). Within $\text{RCA}_0$, let $\hat{A}$ and $\hat{B}$ be complete separable metric spaces. A (code for a) continuous partial function $\phi$ from $\hat{A}$ to $\hat{B}$ is a set of quintuples $\Phi \subseteq \mathbb{N} \times A \times \mathbb{Q}^+ \times B \times \mathbb{Q}^+$ which is required to have certain properties. We write $(a, r)\Phi(b, s)$ as an abbreviation for $\exists n ((n, a, r, b, s) \in \Phi)$. The properties which we require are:

1. if $(a, r)\Phi(b, s)$ and $(a, r)\Phi(b', s')$, then $d(b, b') \leq s + s'$;
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Based on a construction by D. Normann, U. Kohlenbach shows that these two definitions are equivalent in a weak higher-order system based on the well-known weak König’s lemma (WKL).

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(\forall \varepsilon > 0, x \in [0, 1]) (\exists \delta > 0) (\forall y \in [0, 1]) (|x - y| < \delta \rightarrow |f(x) - f(y)| < \varepsilon).
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Problem solved: using codes as in Def. II.6.1 or plain \(\varepsilon\)-\(\delta\)-continuity yields the ‘same theorems’, assuming WKL.
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**Theorem (Arzelà, 1885)**

*Let* $f_n : ([0, 1] \times \mathbb{N}) \to \mathbb{R}$ *be a sequence such that*

1. Each $f_n$ is Riemann integrable on $[0, 1]$.
2. There is $M > 0$ such that $(\forall n \in \mathbb{N}, x \in [0, 1])(|f_n(x)| \leq M)$.
3. $\lim_{n \to \infty} f_n = f$ exists and is Riemann integrable.

*Then* $\lim_{n \to \infty} \int_0^1 f_n(x)dx = \int_0^1 f(x)dx$. 
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Formulated with codes in \( L_2 \), this theorem falls in the ‘weak’ range.
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Formulated with codes in \( L_2 \), this theorem falls in the ‘weak’ range.

Formulated without codes, this theorem is at the very top of the ‘medium’ range (near \( Z_2 \)), far beyond the usual range of RM.
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See arxiv: Normann-Sanders, *On the uncountability of $\mathbb{R}$*. 
Intermediate conclusion

If one wishes to study mathematical theorems as they stand, coding in $L^2$ plays the following role:

Coding continuous functions in $L^2$ is OK, following the work of Normann and Kohlenbach.

Coding Riemann integrable functions (=continuous AE and bounded) in $L^2$ is not OK, following the work of Normann-Sanders.

The difference between 'codes' or 'no codes' for Riemann integrable functions can be huge, as shown by Arzela's convergence theorem.

To properly study discontinuous functions, Kohlenbach has proposed higher-order RM involving all finite types.

The language $L_\omega$ has variables for $n \in \mathbb{N}$, $f : \mathbb{N} \rightarrow \mathbb{N}$, $Y : \mathbb{N} \rightarrow \mathbb{N}$, $F : \mathbb{R} \rightarrow \mathbb{R}$, $G : (\mathbb{R} \rightarrow \mathbb{R}) \rightarrow \mathbb{R}$, ...

Higher-order RM is not the full answer, as our answer to Q3 shows.
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Q3: are ‘the’ minimal axioms always unique?
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$\text{PIT}_o$ is one of the first ‘local-global principles’.

**Theorem (PIT}_o, Pincherle, 1885)**

An \textit{locally bounded function on }$2^\mathbb{N}$\textit{ is bounded.}
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1. Assuming a fragment of countable choice, we have \( \text{WKL} \leftrightarrow \text{PIT}_o \), i.e. \( \text{PIT}_o \) is in the weak range.
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No unique/unambiguous minimal collection of axioms!
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In RM, an open set is given by a union of basic open balls \(\bigcup_{n \in \mathbb{N}} (a_n, b_n)\).
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In RM, an open set is given by a union of basic open balls $\bigcup_{n \in \mathbb{N}} (a_n, b_n)$.

Following Kreuzer and others, we have studied open sets in $\mathbb{R}$ via (third-order) characteristic functions. The following thms then behave in the same way as $\text{PIT}_o$:

1. Urysohn lemma
2. Tietze extension theorem
3. Cantor-Bendixson theorem
4. Baire-Category theorem
5. ...
Intermediate conclusion II

Our answers to Q1 and Q3 have yielded the following:
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Coding in $\mathcal{L}_2$ can change the logical strength of thems involving Riemann integrable functions, unacceptable from the pov of RM. Switching to $\mathcal{L}_\omega$ and Kohlenbach's higher-order RM seems to create other problems involving minimal axioms and countable choice. Our hubris: everything seems wrong about RM. Our catharsis: the answer to Q2 shows that all these problems go away.

The aim of RM is: to find the minimal axioms necessary for proving a theorem of ordinary mathematics.

(Q2) What scale does 'minimal' refer to and why choose that one?
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The aim of RM is: to find the minimal axioms necessary for proving a theorem of ordinary mathematics.

(Q2) What scale does ‘minimal’ refer to and why choose that one?
It is striking that a great many foundational theories are linearly ordered by [consistency strength] $\prec$. Of course it is possible to construct pairs of artificial theories which are incomparable under $\prec$. However, this is not the case for the “natural” or non-artificial theories which are usually regarded as significant in the foundations of mathematics.

(Simpson, Gödel Centennial Volume; also: Koelner, Burgess, Friedman,...)
Gödel hierarchy

= ‘comprehension’ hierarchy

MORE sets exist

\[ \uparrow \]

\[ \begin{align*}
\text{strong} & : \\
& \begin{align*}
& \begin{align*}
& \vdots \\
& \text{large cardinals} \\
& \vdots \\
& \text{ZFC} \\
& \text{ZC} \ (\text{Zermelo set theory}) \\
& \text{simple type theory}
\end{align*} \\
& \begin{align*}
& \vdots \\
& Z_2 \ (\text{second-order arithmetic}) \\
& \vdots \\
& \begin{align*}
& II_1^{1}-\text{CA}_0 \ (\text{comprehension for } II_1^{1}\text{-formulas}) \\
& II_2^{1}-\text{CA}_0 \ (\text{comprehension for } II_2^{1}\text{-formulas}) \\
& \text{ATR}_0 \ (\text{arithmetical transfinite recursion}) \\
& \text{ACA}_0 \ (\text{arithmetical comprehension})
\end{align*} \\
& \begin{align*}
& \vdots \\
& \text{WKL}_0 \ (\text{weak König’s lemma}) \\
& \text{RCA}_0 \ (\text{recursive comprehension}) \\
& \text{PRA} \ (\text{primitive recursive arithmetic}) \\
& \text{bounded arithmetic}
\end{align*}
\end{align*}
\end{align*}
\end{align*}
\right)

\[ \downarrow \]

LESS sets exist
Gödel hierarchy

strong

Zermelo-Fraenkel set theory with choice
aka ‘the’ foundation of mathematics

medium

\( Z_2 \) (second-order arithmetic)
\( \Pi^1_2\)-CA\(_0\) (comprehension for \( \Pi^1_2\)-formulas)
\( \Pi^1_1\)-CA\(_0\) (comprehension for \( \Pi^1_1\)-formulas)
\( \text{ATR}_0 \) (arithmetical transfinite recursion)
\( \text{ACA}_0 \) (arithmetical comprehension)

weak

\( \text{WKL}_0 \) (weak König’s lemma)
\( \text{RCA}_0 \) (recursive comprehension)
\( \text{PRA} \) (primitive recursive arithmetic)
bounded arithmetic
## Gödel hierarchy

### strong

Zermelo-Fraenkel set theory with choice  
aka ‘the’ foundation of mathematics

Hilbert-Bernays’s *Grundlagen der Mathematik*

### medium

\[ \vdash \]
- \( Z_2 \) (second-order arithmetic)
- \( \Pi^1_1\text{-CA}_0 \) (comprehension for \( \Pi^1_1 \)-formulas)
- \( \Pi^1_2\text{-CA}_0 \) (comprehension for \( \Pi^1_2 \)-formulas)
- \( \text{ATR}_0 \) (arithmetical transfinite recursion)
- \( \text{ACA}_0 \) (arithmetical comprehension)

### weak

- \( \text{WKL}_0 \) (weak König's lemma)
- \( \text{RCA}_0 \) (recursive comprehension)
- PRA (primitive recursive arithmetic)
- bounded arithmetic

- \( \vdash \)
- large cardinals
- \( \vdash \)
- ZFC
- ZC (Zermelo set theory)
- simple type theory

- \( \vdash \)
- Received view: natural/important systems form linear Gödel hierarchy
- and 80/90% of ordinary mathematics is provable in ACA\( _0^{1-\Pi} \).
Gödel hierarchy

**strong**

Zermelo-Fraenkel set theory with choice
aka ‘the’ foundation of mathematics

Hilbert-Bernays’s *Grundlagen der Mathematik*

**medium**

Russell-Weyl-Feferman
predicative mathematics

**weak**

WKL₀ (weak König’s lemma)
RCA₀ (recursive comprehension)
PRADA (primitive recursive arithmetic)
bounded arithmetic

\[\vdots\]

large cardinals

\[\vdots\]

ZFC

ZC (Zermelo set theory)
simple type theory

\[\vdots\]

\[Z₂\text{ (second-order arithmetic)}\]

\[\vdots\]

\[IΠ²¹⁻CA₀\text{ (comprehension for } IΠ²¹\text{-formulas)}\]

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ATR₀ (arithmetical transfinite recursion)

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\[\vdots\]
Gödel hierarchy

strong

Zermelo-Fraenkel set theory with choice
aka ‘the’ foundation of mathematics

Hilbert-Bernays’s Grundlagen der Mathematik

medium

Russell-Weyl-Feferman predicative mathematics

The ‘Big Five’ of Reverse Mathematics

weak

\begin{align*}
\text{large cardinals} \\
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\text{\begin{align*}
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\end{align*}}
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### Gödel hierarchy

<table>
<thead>
<tr>
<th>Level</th>
<th>Systems</th>
</tr>
</thead>
<tbody>
<tr>
<td>strong</td>
<td>Zermelo-Fraenkel set theory with choice (aka ‘the’ foundation of mathematics)</td>
</tr>
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<td></td>
<td>Hilbert-Bernays’s <em>Grundlagen der Mathematik</em></td>
</tr>
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</tr>
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Received view: natural/important systems form linear Gödel hierarchy and 80/90% of ordinary mathematics is provable in ACA$_0$/$\Pi^1_1$-$CA_0$.}

Hilbert-Bernays’s *Grundlagen der Mathematik*
Gödel hierarchy

strong

Zermelo-Fraenkel set theory with choice
aka ‘the’ foundation of mathematics

Hilbert-Bernays’s Grundlagen
der Mathematik

medium

Russell-Weyl-Feferman
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In *Grundlagen der Mathematik*, Hilbert and Bernays formalise (a lot of) mathematics in a logical system $H$.
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System $H$ makes (essential) use of third-order parameters, but is ‘more second-order’ than previous systems (with Ackermann).
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$H$ inspired second-order arithmetic $Z_2$ based on comprehension:

$$(\exists X \subset \mathbb{N})(\forall n \in \mathbb{N})(n \in X \leftrightarrow \varphi(n))$$

for any formula $\varphi(n)$ in $L_2$, language of $Z_2$. 

Indeed, the following is (explicitly) introduced in $H$:

$$(\exists n \in \mathbb{N})(f(n) = 0) \rightarrow f(\mu(f)) = 0 \text{ (Feferman's $\mu$)}$$

yielding arithmetical comprehension as in ACA$_0$.

Similarly:

$\nu$-functional produces witness to $$(\exists f : \mathbb{N} \rightarrow \mathbb{N})A(f),$$

yielding $Z_2$. 

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\( Z_2^\omega \) is based on \textbf{comprehension} as follows:

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(\exists f : \mathbb{N} \to \mathbb{N})A(f) \leftrightarrow A(\nu_{k+1}g.A(g)) \quad (\ast)
\]

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for any third-order $Y : \mathbb{N}^\mathbb{N} \to \mathbb{N}$. 

E is called Kleene’s $\exists_3$.
Comprehension by any other name

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Connection: $Z^2 \equiv L^2 Z^\omega_2 \equiv L^2 Z^\Omega_2$.

Note 3rd vs 4th order!
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Incomprehensible!

Recall that $Z_2 \equiv_{L_2} Z^\omega_2 \equiv_{L_2} Z^\Omega_2$. 
Incomprehensible!

Recall that $Z_2 \equiv_{L_2} Z_2 \equiv_{L_2} Z_2^\Omega$. The following third-order theorems are provable in $Z_2^\Omega$, but not in $Z_2$. 

---

1. Arzelà's convergence theorem for Riemann integral (1885).
2. A countably-compact metric space $(0,1,d)$ is separable.
3. Baire category theorem (open sets as characteristic functions).
4. There is a function $f: \mathbb{R} \to \mathbb{R}$ not in Baire class 2.
5. Baire characterisation theorem for Baire class 1.
6. Heine-Borel/Vitali/Lindelöf for uncountable coverings.
7. Basic Lebesgue measure/integral and gauge integral.
8. Unordered sums $\sum_{x \in \mathbb{R}} f(x)$ are countable (E.H. Moore).
9. Convergence theorems for nets indexed by $\mathbb{N}$ (Moore-Smith).
10. The uncountability of $\mathbb{R}$: there is no injection (or bijection) from $[0,1]$ to $\mathbb{N}$ (Cantor, 1874).
11. Basic RM theorems with usual definition of countable set.
Incomprehensible!

Recall that $Z_2 \equiv_{L_2} Z_2^\omega \equiv_{L_2} Z_2^\Omega$. The following \textit{third-order} theorems are provable in $Z_2^\Omega$, but not in $Z_2^\omega$.

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Incomprehensible!

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10. The \textit{uncountability of $\mathbb{R}$}: there is no injection (or bijection) from $[0, 1]$ to $\mathbb{N}$ (Cantor, 1874).
11. Basic RM theorems with usual definition of \textit{countable set}. 

\textit{Incomprehensible!}
Uncountability of \( \mathbb{R} \)
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**Theorem (NIN, see Kunen)**

For $Y : [0, 1] \rightarrow \mathbb{N}$, there are $x, y \in [0, 1]$ s.t. $x \neq y \land Y(x) = Y(y)$

**Theorem (NBI, see Hrbacek-Jech)**

For $Y : [0, 1] \rightarrow \mathbb{N}$, there are distinct $x, y \in [0, 1]$ such that $Y(x) = Y(y)$ OR there is $n \in \mathbb{N}$ with $(\forall x \in [0, 1])(Y(x) \neq n)$. 
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These are provable in $\mathcal{Z}_2^\omega$ but not in $\mathcal{Z}_2^\omega$ (and the weakest such).
Two nice observations about the uncountability of $\mathbb{R}$

Firstly, $Z_2^\omega + \neg \text{NBI}$ proves $Z_2$ and is consistent.
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In contrast to the modern era, Weierstrass changed his mind in light of Cantor’s work...
Countable sets versus sets that are countable

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Explosion: $\Pi^1_2$-CA$_0$ follows from item (a) plus $\Pi^1_1$-CA$_\omega$$_0$.

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(a) A countable subset of $[0,1]$ has a supremum (Bolzano-Weierstrass).
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**PROBLEM:** hundreds of intuitively weak third-order theorems are classified as rather strong qua third-order comprehension, i.e. not provable in $\mathbb{Z}_2^{\omega}$ and provable in $\mathbb{Z}_2^{\Omega}$, for $\mathbb{Z}_2 \equiv_{L_2} \mathbb{Z}_2^{\omega} \equiv_{L_2} \mathbb{Z}_2^{\Omega}$. 
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**CAUSE:** comprehension functionals (like $\mu, \nu_n, \exists^3$) are discontinuous (or: normal).
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SOLUTION: split the hierarchy below $Z_2^\Omega$ in normal and non-normal part.

Normal part with hierarchy $\Pi_k^1$-CA$_0^\omega$ and discontinuous functionals $\nu_k$. (Kohlenbach)
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Brouwer and continuity to the rescue

L.E.J. Brouwer is (in)famous for his *intuitionism*. 

\[
\text{Definition (NFP, 1970, Kreisel-Troelstra)}
\]

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\forall f \in \mathbb{N} \quad \exists n \in \mathbb{N} \quad \forall A \quad A(f_n) \rightarrow \exists \gamma \in K_0 \quad \forall f \in \mathbb{N} \quad A(f_\gamma(f))
\]

Note that \(f_n\) is the finite sequence \(\langle f(0), f(1), \ldots, f(n-1) \rangle\). NFP expresses that there are (many) continuous choice functions.
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**Definition (NFP, 1970, Kreisel-Troelstra)**

For any formula $A$, we have

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1) Many non-normal theorems (Heine-Borel, Lindelöf, monotone convergence theorem for nets, ...) are equivalent to natural fragments of NFP.

2) The equivalences from 1) map to the Big Five equivalences, under the canonical embedding of HOA in SOA.
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The second item reminds one of Plato’s allegory of the cave.
Plato and his -ism

Plato is well-known in (foundations of) mathematics for his eponymous philosophy platonism, i.e. the theory that mathematical objects are objective, timeless entities, independent of the physical world and the symbols that represent them. Plato's allegory of the cave provides a powerful visual: We can only know reflections/shadows/... of ideal objects.
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Big Five and equivalences

ECF

ECF is canonical embedding of HOA into SOA (Kleene-Kreisel).
The Big Five as a reflection
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- $\Pi^1_1$-CA$_0$
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- ACA$_0$
- WKL$_0$ $\iff$ Dini’s theorem.
  $\iff$ countable Heine-Borel compactness
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  - thms about closed sets (as countable unions)
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SECOND-ORDER arithmetic
The Big Five as a reflection

\[ \text{RCA}_0 \] proves \( \Delta^0_1 \)-comprehension

\[ \text{WKL}_0 \] \iff \text{Dini’s theorem.}
\iff \text{countable Heine-Borel compactness}
\iff \text{Riemann int. thms}

\[ \text{ATR}_0 \] \iff \text{perfect set theorem}
\iff \text{range of } f : \mathbb{N} \to \mathbb{N} \text{ exists}
\iff \text{Monotone conv. thm}
\iff \text{Ascoli-Arzela}
\iff \text{thms about closed sets (as countable unions)}

\[ \text{ACA}_0 \] \iff \text{Monotone conv. thm}
\iff \text{Ascoli-Arzela}
\iff \text{thms about closed sets (as countable unions)}

\[ \Pi^1_1 \text{-CA}_0 \] \iff \text{Cantor-Bendixson thm}

\[ \text{WKL}_1 \] \iff \text{countable Heine-Borel compactness: HBU}
\iff \text{gauge integral thms}
\iff \text{range of } Y : \mathbb{N} \to \mathbb{N} \text{ exists}
\iff \text{Monotone conv. thm for nets}
\iff \text{Ascoli-Arzela for nets}
\iff \text{thms about closed sets (as uncountable unions)}

\[ \text{ECF} \] replaces uncountable objects by countable representations/RM-codes

\[ \Sigma^0_1 \text{-TR} \] \iff \text{Dini’s theorem for nets.}
\iff \text{uncountable Heine-Borel compactness: HBU}
\iff \text{gauge integral thms}
\iff \text{range of } Y : \mathbb{N} \to \mathbb{N} \text{ exists}
\iff \text{Monotone conv. thm for nets}
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ECF converts right-hand side to left-hand side, including equivalences!
The Big Five as a reflection

\[ \Pi^1_1 - \text{CA}_0 \leftrightarrow \text{Cantor-Bendixson thm} \]
\[ \text{ATR}_0 \leftrightarrow \text{perfect set theorem} \]
\[ \leftrightarrow \text{range of } f : \mathbb{N} \rightarrow \mathbb{N} \text{ exists} \]
\[ \leftrightarrow \text{Monotone conv. thm} \]
\[ \leftrightarrow \text{Ascoli-Arzela} \]
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\[ \text{WKL}_0 \leftrightarrow \text{Dini’s theorem.} \]
\[ \leftrightarrow \text{countable Heine-Borel compactness} \]
\[ \leftrightarrow \text{Riemann int. thms} \]
\[ \text{RCA}_0 \text{ proves } \Delta^0_1 - \text{comprehension} \]

\[ \text{SECOND-ORDER arithmetic} \]

\[ \text{BOOT}_2 \]
\[ \Sigma - \text{TR} \]
\[ \text{BOOT} \]
\[ \text{WKL}^1 \]

\[ \text{RCA}_0^\omega \text{ plus a fragment of countable choice} \]

\[ \text{HIGHER-ORDER arithmetic} \]
The Big Five as a reflection

\[
\begin{align*}
&II_1^1 - CA_0 \iff \text{Cantor-Bendixson thm} \\
&ATR_0 \iff \text{perfect set theorem} \\
&A CA_0 \iff \text{range of } f : \mathbb{N} \to \mathbb{N} \text{ exists} \\
&W KL_0 \iff \text{Dini's theorem.} \\
&R CA_0 \iff \Delta_1^0 \text{-comprehension} \\
\end{align*}
\]
The Big Five as a reflection

$\Pi^1_1$-CA$_0 \iff$ Cantor-Bendixson thm

$\text{ATR}_0 \iff$ perfect set theorem

$\iff$ range of $f : \mathbb{N} \to \mathbb{N}$ exists

$\iff$ Monotone conv. thm

$\iff$ Ascoli-Arzela

$\iff$ thms about closed sets (as countable unions)

$\text{WKL}_0 \iff$ Dini’s theorem.

$\iff$ countable Heine-Borel compactness

$\iff$ Riemann int. thms

$\text{RCA}_0$ proves $\Delta^0_1$-comprehension

SECOND-ORDER arithmetic

$\text{BOOT}_2$

$\Sigma$-TR

$\iff$ range of $Y : \mathbb{N}^\mathbb{N} \to \mathbb{N}$ exists

$\iff$ Mon. conv. thm for nets

$\iff$ Ascoli-Arzela for nets

$\iff$ thms about closed sets (as uncountable unions)

$\text{WKL}^1 \iff$ Dini’s theorem for nets.

$\iff$ uncountable Heine-Borel compactness: HBU

$\iff$ gauge integral thms

$\text{RCA}_\omega^\omega$ plus a fragment of countable choice

HIGHER-ORDER arithmetic
The Big Five as a reflection

\[ \mathcal{II}_1^{1}\text{-CA}_0 \leftrightarrow \text{Cantor-Bendixson thm} \]

\[ \mathcal{ATR}_0 \leftrightarrow \text{perfect set theorem} \]

\[ \mathcal{ACA}_0 \leftrightarrow \begin{align*} &\text{range of } f : \mathbb{N} \to \mathbb{N} \text{ exists} \\
&\text{Monotone conv. thm} \\
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\[ \mathcal{WKL}_0 \leftrightarrow \begin{align*} &\text{Dini’s theorem.} \\
&\text{countable Heine-Borel compactness} \\
&\text{Riemann int. thms} \end{align*} \]

\[ \mathcal{RCA}_0 \text{ proves } \mathcal{\Delta}^{1}_0\text{-comprehension} \]

SECOND-ORDER arithmetic

Cantor-Bendixson thm

\[ \mathcal{BOOT}_2 \leftrightarrow \text{(uncountable unions)} \]

\[ \mathcal{\Sigma-TR} \]

\[ \mathcal{\Sigma-TR} \leftrightarrow \begin{align*} &\text{range of } Y : \mathbb{N}^\mathbb{N} \to \mathbb{N} \text{ exists} \\
&\text{Mon. conv. thm for nets} \\
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\[ \mathcal{WKL}^1 \leftrightarrow \begin{align*} &\text{Dini’s theorem for nets.} \\
&\text{uncountable Heine-Borel compactness: HBU} \\
&\text{gauge integral thms} \end{align*} \]

\[ \mathcal{RCA}^\omega \leftrightarrow \begin{align*} &\text{plus a fragment of countable choice} \\
&\text{HIGHER-ORDER arithmetic} \end{align*} \]

\[ \mathcal{ECF} \text{ replaces uncountable objects by countable representations/RM-codes} \]

\[ \text{←− ECF} \]

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The Big Five as a reflection

- $II^1_1-CA_0 \iff$ Cantor-Bendixson thm
- $ATR_0 \iff$ perfect set theorem
  - $\iff$ range of $f : \mathbb{N} \to \mathbb{N}$ exists
  - $\iff$ Monotone conv. thm
  - $\iff$ Ascoli-Arzela
  - $\iff$ thms about closed sets (as countable unions)
- $ACA_0 \iff$ Dini’s theorem.
  - $\iff$ countable Heine-Borel compactness
  - $\iff$ Riemann int. thms
- $WKL_0 \iff$ Dini’s theorem for nets.
  - $\iff$ countable Heine-Borel compactness
  - $\iff$ Riemann int. thms
- $RCA_0$ proves $\Delta^0_1$-comprehension

Cantor-Bendixson thm

- $BOOT_2 \iff$ (uncountable unions)
- $\Sigma$-TR $\iff$ perfect set thm (idem)
  - $\iff$ range of $Y : \mathbb{N}^\mathbb{N} \to \mathbb{N}$ exists
- $BOOT \iff$ Mon. conv. thm for nets
  - $\iff$ Ascoli-Arzela for nets
  - $\iff$ thms about closed sets (as uncountable unions)
- $WKL^1 \iff$ Dini’s theorem for nets.
  - $\iff$ uncountable Heine-Borel compactness: HBU
  - $\iff$ gauge integral thms
- $RCA^\omega_0$ plus a fragment of countable choice

SECOND-ORDER arithmetic

HIGHER-ORDER arithmetic

SECOND-ORDER arithmetic

HIGHER-ORDER arithmetic
The Big Five as a reflection

ECF replaces uncountable objects by countable representations/RM-codes

\( \mathbb{II}_1^{1} \text{-CA}_0 \leftrightarrow \text{Cantor-Bendixson thm} \)

\( \text{ATR}_0 \leftrightarrow \text{perfect set theorem} \)

\( \text{ACA}_0 \leftrightarrow \text{range of } f : \mathbb{N} \to \mathbb{N} \text{ exists} \)
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SECOND-ORDER arithmetic

\( \text{ECF} \rightarrow \)

\( \text{BOOT}_2 \leftrightarrow \text{(uncountable unions)} \)

\( \Sigma^0_1 \text{-TR} \leftrightarrow \text{perfect set thm (idem)} \)

\( \text{BOOT} \leftrightarrow \text{range of } Y : \mathbb{N}^\mathbb{N} \to \mathbb{N} \text{ exists} \)
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HIGHER-ORDER arithmetic
The Big Five as a reflection

**ECF** replaces **uncountable** objects by **countable** representations/RM-codes
**ECF** converts right-hand side to left-hand side, **including equivalences!**
Foundations/philosophy of mathematics

One can (and people probably will) argue forever which ‘-ism’ is the true foundations/philosophy of mathematics.
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One could also take a hint from the exact sciences (to which math technically belongs) and try to find evidence in support of one’s viewpoint.

I present the previous picture as evidence supporting Platonism.
Conclusion
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To properly study discontinuous functions, one adopts Kohlenbach’s higher-order RM. This ‘normal’ scale however classifies ‘intuitively weak’ theorems as ‘rather strong’, including the uncountability of $\mathbb{R}$. 
Conclusion

Coding in $L_2$ is not bad *per se*: it works for *continuous* functions, but is a bad idea for *discontinuous* functions from the pov of RM. This is witnessed by basic theorems, like *Arzela’s convergence thm* for the Riemann integral.

To properly study discontinuous functions, one adopts Kohlenbach’s *higher-order RM*. This ‘normal’ scale however classifies ‘intuitively weak’ theorems as ‘rather strong’, including the *uncountability of $\mathbb{R}$*.

To solve this problem, one adopts the *complimentary non-normal* scale based on classically valid *continuity* axioms (NFP) from Brouwer’s intuitionistic mathematics.

In the spirit of Plato’s cave, the Big Five of RM are a reflection of the non-normal scale under Kleene-Kreisel’s ECF.
Final Thoughts
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Thank you for your attention!

Any (content) questions?
Raphael’s Annotated School of Athens

Mind tricks don’t work on me! Only ideal objects!

These are not the Big Five you are looking for?