Discussion minutes: UConn Logic Group: November 21, 2008

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The session consisted of a discussion of Greg Restall's (1994) paper: "Arithmetic and Truth in Łukasiewicz's Infinitely Valued Logic"

Reed guided us through the main result of the paper that $Ł_\infty^{\#T}$ is ω-inconsistent. Let A_0 be the sentence

$$\neg\forall x\exists y(R(x+1,y)\wedge T(y))$$

and $A_{n+1} = A_0 \circ A_0 \circ ... \circ A_0 = A_0^{n+1}$ be the $(n+1)$-fold fusion of A_0, where $R(x,y)$ is a (formula coding the) predicate that holds if and only if y is the Gödel number of A_0^x. (The existence of such a predicate and coding formula will be discussed in our next meeting.) We wish to show that A_0 witnesses the ω-inconsistency of $Ł_\infty^{\#T}$.

First we show that it cannot be that $\nu(A_0) \neq 0$ (where 0 is the designated *true* value). Suppose $\nu(A_0) \neq 0$ and we derive a contradiction as follows:

> By evaluation of the fusion operator, $\nu(A_n) = 1$ for some $n > 0$, so fix such n.
>
> For $y = \ulcorner A_n \urcorner$, we have $\nu(T(y)) = \nu(A_n) = 1$.
>
> Thus, $\nu(R(\underline{n},y)\wedge T(y)) = \max\{\nu(R(\underline{n},y)), \nu(T(y))\} = 1$.
>
> For $y \neq \ulcorner A_n \urcorner$, we have $\nu(R(\underline{n},y)) = 1$.
>
> Thus, $\nu(R(\underline{n},y)\wedge T(y)) = \max\{\nu(R(\underline{n},y)), \nu(T(y))\} = 1$.
>
> So, $\nu(\exists y(R(\underline{n},y)\wedge T(y)) = \inf\{\nu(R(\underline{n},y)\wedge T(y)) \mid y \in M\} = 1$.
>
> So, $\nu(\forall x\exists y(R(x+1,y)\wedge T(y)) = \sup\{\nu(R(x+1,y)\wedge T(y)) \mid x \in M\} = 1$.
>
> However, then $\nu(A_0) = \nu(\neg\forall x\exists y(R(x+1,y)\wedge T(y)) = 1-1 = 0$, which is a contradiction.

The only possibility then, is that $\nu(A_0) = 0$. Let $\varphi(x)$ be the sentence $\exists y(R(x+1,y)\wedge T(y))$ and notice that A_0 is $\neg\forall x\varphi(x)$. We now reason as follows.

Since $\nu(A_0) = 0$ holds in an arbitrary model of $Ł_\infty^{\#T}$, $Ł_\infty^{\#T} \models A_0$.

Therefore, $Ł_\infty^{\#T} \models \neg\forall x\varphi(x)$.

By evaluation of the fusion operator, $\nu(A_{n+1}) = 0$ for all $n \in \mathbb{N}$.

Fix $n \in \mathbb{N}$ and let $y = \ulcorner A_{n+1} \urcorner$.

Then $\nu(R(\underline{n+1}, y)) = 0$ and $\nu(T(y)) = \nu(A_{n+1}) = 0$.

Thus, $\nu(\exists y(R(\underline{n+1}, y) \wedge T(y))) = 0$ for all $n \in \mathbb{N}$.

In other words, $\nu(\varphi(\underline{n})) = 0$ for all $n \in \mathbb{N}$.

Since this holds in an arbitrary model of $Ł_\infty^{\#T}$, $Ł_\infty^{\#T} \models \varphi(\underline{n})$ for all $n \in \mathbb{N}$.

Thus, $Ł_\infty^{\#T}$ is ω-inconsistent.

The following points were raised in discussion:

1. Reed remarked that Restall's argument relies on $\nu(A_\circ) \neq 0 \Rightarrow \nu(A_n) = 1$ for some $n \in \mathbb{N}$, and for this to be right we need it that $[0, 1]$ has the following Archimedean property: if $0 < r$, then there is some $n \in \mathbb{N}$ such that $n \cdot r \geq 1$. Given this, Reed wondered, which is worse: ω-inconsistency or nonstandard real numbers?

    Jc recalled that Gupta has an ω-inconsistent model of arithmetic. Lionel then offered that Gupta may not be affected by Restall's criticism since Gupta does not have a standard model in his theory. When asked by Jc whether the existence of nonstandard real number would make this arithmetic theory ω-consistent, Reed answered that it is not clear but that this argument would not work to show ω-inconsistency.

2. Scott wondered what could count as independent motivation for the Łukasiewicz semantics.

    Jc responded that if one thinks that there exists a fully disquotational T-predicate in our language, then you have motivation for Łukasiewicz semantics. Further, if classical logic is an extension of Ł, then you can build other T-predicates to capture other kinds of truth.

3. Reed asked if one is willing to have non-classical truth values for a language, why should mathematics be of a different status (i.e. have only classical values)?

   > Jc responded that he was happy to say that mathematics ought to be defined in whichever way classical mathematicians regard as correct. Since mathematicians (by in large) maintain that mathematics is classical, it's logic ought to be classical. However, beyond the mathematics building, there is another (there are other?) T-predicate(s?) (which is (are?) not used in classical mathematics). Vagueness (for example) motivates the move to a non-clasical semantics. That is, while mathematics doesn't require non-classicality, other fields do.

4. Reed asked what is the special role of mathematics, such that it remains outside the problems which motivate non-classical logics.

   > Jc responded that perhaps things like Russell's paradox will ultimately motivate a non-classical approach. Perhaps tomorrow mathematics will decide to go non-classical...

5. Reed remarked that there are some parts of mathematics (e.g. the continuum hypothesis) which do receive a third value (of a sort), i.e. 'independent'. So, in a sense mathematicians do use three values. Jc then offered that the non-classical logics for truth are exactly those which are used to capture what is at issue here.

6. Lionel suggested that we refer back to how Restall addresses the issue:

   > "We need not take classical logic as the best theory of inference, while agreeing that it is correct for reasoning about numbers. This need not be true for other predicates we may wish to add to the language. A predicate added to the language is not bound to be classical." (Restall p. 4)

   Then on Restall's picture, Lionel suggested, it sounds like there are some fragments of the non-classical language that are classical (i.e. fragments in which LEM holds). Given this, we might ask what counts as "logic"? LEM is perhaps a mathematical principle, and we ought therefore to use it (in some contexts) but the fragment in which we use LEM would not then be "logic".

> Jc remarked that one problem with this sort of view will be that there is no way of saying (within a non-classical language e.g. FDE) "this fragment is classical".

7. Aaron wondered whether the logic of mathematics need be strong or weak Kleene.

> In response to Michael asking about the difference between the two, it was clarified that in both weak and strong Kleene a disjunction where one disjunct is false, and the other neither, itself recieves the value neither. Where a disjunction has one disjunct true and the other neither, the disjunction is itself neither in weak Kleene, but true in strong Kleene. This latter result gives a reason for preferring strong Kleene as the logic of mathematics.

8. Lionel wondered whether it was correct to say of $A_\circ$ that if it is not true, then it is true.

> Reed clarified that $A_\circ$ says that, 'everything in the sequence is not false'.

9. Lionel wondered whether it was interesting to note that we're reasoning classically in working towards building (considering?) a non-classical logic of arithmetic.

> Jc responded that we're using classical set-theory to set up a non-classical language, which seems to make sense, given our understanding of the nature of mathematics. That is, if we're only specifying the validity relation we could have a proper part of the language which is classical, without the whole language being classical.

10. Marcus commented that while we've reasoned classically in order to get a non-classical model, once we have that non-classical model, we would seem to lose classicality since our model is non-classical.

Similarly, Reed remarked that perhaps the fixed point provides justification inside the formal theory for the move from truth to truth in the language, but wondered if we obliterate classicality in the process. Jc responded that we can use a proper fragment of the language (in which the classical consequence relation holds) to justify the broader language. Reed responded that this would be building a T-predicate in a language in which you can't justify the T-predicate within the proper fragment of the language. But again, perhaps the fixed point saves the day here.