Ł$_\aleph$ Fuzzy Logic, Part II

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Quantified Ł$_\aleph$ Syntax and Semantics

We work in a formal language the syntax of which consists of. . .

1. variables x, y, z
2. constants a, b, c
3. function signs f, g, h
4. non-logical predicates F, G, H
5. identity predicate $=$
6. connectives $\neg$, $\vee$, $\wedge$, $\rightarrow$
7. quantifiers $\exists$, $\forall$
8. brackets ( )

The terms of the language include the variables and constants, as well as any string $ft_1 \ldots t_n$ where f is an n-ary function sign and $t_1 \ldots t_n$ are n-many terms.

The formation rules for formulas are. . .

1. If F is an n-arypredicate and $t_1 \ldots t_n$ are terms, then $Ft_1 \ldots t_n$ is an (atomic) formula (in place of $= t_1 t_2$ we will typically write $t_1 = t_2$).
2. If A is a formula, then $\neg A$ is a formula.
3. If A and B are formulas, so too are $A \wedge B$, $A \vee B$, $A \rightarrow B$.
4. If A is a formula and x is a variable, then $\exists x A$ and $\forall x A$ are formulas.
5. Nothing is a formula except by (1)-(5)

A quantified $Ł_{\aleph}$ interpretation is a pair $\mathfrak{I} = \langle D, \nu \rangle$ comprising a domain D and a valuation function ν. The non-logical vocabulary is interpreted as follows.

$\nu(a) \in D$ for each constant a.

$\nu(f) : D^n \to D$ for each n-ary function sign f.

$\nu(F) : D^n \to [0, 1]$ for each n-ary predicate F.

The identity predicate raises interesting questions in this context. For now, let every interpretation assign to the identity predicate the (classical) function $\nu(=)$ such that $\nu(=)\langle d_1, d_2 \rangle = 1$ iff d_1 and d_2 are the same object in the domain, otherwise $\nu(=)\langle d_1, d_2 \rangle = 0$. A variable assignment σ maps variables as follows.

$\sigma(x) \in D$ for each variable x.

The denotation of term t under $\mathfrak{I} = \langle D, \nu \rangle$ and σ is $\delta^{\sigma}_{\mathfrak{I}}(t)$ defined as follows.

$$\delta^{\sigma}_{\mathfrak{I}}(t) = \begin{cases} \nu(t) \text{ if } t \text{ is a constant} \\ \sigma(t) \text{ if } t \text{ is a variable} \\ \nu(f)\langle \delta^{\sigma}_{\mathfrak{I}}(t_1), \ldots, \delta^{\sigma}_{\mathfrak{I}}(t_n) \rangle \text{ if } t \text{ is of the form } ft_1 \ldots t_n \end{cases}$$

In what follows, we will adopt the following conventions for the sake of readability. Let $A_{[t/x]}$ be the formula that results from substituting the term t for all instances of the variable x in formula A. Let lub and glb stand for, respectively, the *least upper bound* of a set of numerical values and the *greatest lower bound* of a set of numerical values (where x is a lower bound of the set Y means that $x \leq y$ for all $y \in Y$ and so x is the greatest lower bound of Y if and only if of all the lower bounds of Y it is the greatest, i.e. numerically largest in this case).

Say that variable assignment σ' is an x-*alternative* to assignment σ just in case σ' differs from σ at most in its interpretation of the variable x. I will sometimes just write this as "σ' an x-alt. to σ" to abbreviate even further.

The semantic value assigned to the formula A by the interpretation $\mathfrak{I} = \langle D, \nu \rangle$ paired with variable assignment σ is $\nu^{\sigma}_{\mathfrak{I}}(A)$ calculated as follows.

1. $\nu^{\sigma}_{\mathfrak{I}}(Ft_1 \dots t_n) = \nu(F)\langle \delta^{\sigma}_{\mathfrak{I}}(t_1), \dots, \delta^{\sigma}_{\mathfrak{I}}(t_n) \rangle$ for atomic sentence $Ft_1, \dots t_n$.

2. $\nu^{\sigma}_{\mathfrak{I}}(\neg A) = 1 - \nu^{\sigma}_{\mathfrak{I}}(A)$.

3. $\nu^{\sigma}_{\mathfrak{I}}(A \wedge B) = \min\{\nu^{\sigma}_{\mathfrak{I}}(A), \nu^{\sigma}_{\mathfrak{I}}(B)\}$.

4. $\nu^{\sigma}_{\mathfrak{I}}(A \vee B) = \max\{\nu^{\sigma}_{\mathfrak{I}}(A), \nu^{\sigma}_{\mathfrak{I}}(B)\}$.

5. $\nu^{\sigma}_{\mathfrak{I}}(A \rightarrow B) = \begin{cases} 1 \text{ if } \nu^{\sigma}_{\mathfrak{I}}(A) \leq \nu^{\sigma}_{\mathfrak{I}}(B) \\ 1 - (\nu^{\sigma}_{\mathfrak{I}}(A) - \nu^{\sigma}_{\mathfrak{I}}(B)) \text{ if } \nu^{\sigma}_{\mathfrak{I}}(A) > \nu^{\sigma}_{\mathfrak{I}}(B) \end{cases}$

6. $\nu^{\sigma}_{\mathfrak{I}}(\exists x A) = \mathrm{lub}\{\nu^{\sigma'}_{\mathfrak{I}}(A) \mid \sigma'$ an x-alt. to $\sigma\}$

7. $\nu^{\sigma}_{\mathfrak{I}}(\forall x A) = \mathrm{glb}\{\nu^{\sigma'}_{\mathfrak{I}}(A) \mid \sigma'$ an x-alt. to $\sigma\}$

Much as before, the formula A is satisfied under interpretation $\mathfrak{I} = \langle D, \nu \rangle$ paired with variable assignment σ iff $\nu^{\sigma}_{\mathfrak{I}}(A) = 1$. Similarly, the set of formulas Σ is satisfied under $\mathfrak{I}$ and σ iff $\nu^{\sigma}_{\mathfrak{I}}(B) = 1$ for every $B \in \Sigma$. Hopefully there will be no confusion if I use the same symbol for consequence as I did last week.

Semantic definition of quantified Ł$_{\aleph}$ consequence ($\vDash_{\aleph}$).

> $\Sigma \vDash_{\aleph} A$ iff in every quantified Ł$_{\aleph}$ interpretation $\mathfrak{I} = \langle D, \nu \rangle$ paired with variable assignment σ under which Σ is satisfied, A is satisfied.

We can now look at inferences which hold in this logic. It turns out that most of the non-classical behavior of this logic is due to the propositional fragment.

Some Notable Inferences

Fact: $\models_{\aleph}\ \forall x A \rightarrow A_{[t/x]}$ (for any term t)

Proof: For any interpretation $\mathfrak{I} = \langle D, \nu \rangle$ and variable assignment σ we have $\nu^{\sigma}_{\mathfrak{I}}(\forall x A) = \mathrm{glb}\{\nu^{\sigma'}_{\mathfrak{I}}(A) \mid \sigma' \text{ an } x\text{-alt. to } \sigma\} \leq \nu^{\sigma}_{\mathfrak{I}}(A_{[t/x]})$ for any term t.

Fact: $\models_{\aleph}\ A_{[t/x]} \rightarrow \exists x A$ (for any term t)

Proof: For any interpretation $\mathfrak{I} = \langle D, \nu \rangle$ and variable assignment σ we have that, for any term t, $\nu^{\sigma}_{\mathfrak{I}}(A_{[t/x]}) \leq \mathrm{lub}\{\nu^{\sigma'}_{\mathfrak{I}}(A) \mid \sigma' \text{ an } x\text{-alt. to } \sigma\} = \nu^{\sigma}_{\mathfrak{I}}(\exists x A)$

Fact: $\forall x \neg A \models_{\aleph} \neg \exists x A$ and $\neg \exists x A \models_{\aleph} \forall x \neg A$

Proof: These follow from the fact that in any interpretation $\mathfrak{I} = \langle D, \nu \rangle$ and variable assignment ν we have $\nu^{\sigma}_{\mathfrak{I}}(\forall x \neg A) = \nu^{\sigma}_{\mathfrak{I}}(\neg \exists x A)$. The relevant stages in a proof of *this* fact are as follows. Steps (1),(2),(5),(6) are true by definition.

1. $\nu^{\sigma}_{\mathfrak{I}}(\forall x \neg A) = \mathrm{glb}\{\nu^{\sigma'}_{\mathfrak{I}}(\neg A) \mid \sigma' \text{ an } x\text{-alt. to } \sigma\}$
2. $\mathrm{glb}\{\nu^{\sigma'}_{\mathfrak{I}}(\neg A) \mid \sigma' \text{ an } x\text{-alt. to } \sigma\} = \mathrm{glb}\{1 - \nu^{\sigma'}_{\mathfrak{I}}(A) \mid \sigma' \text{ an } x\text{-alt. to } \sigma\}$
3. $\mathrm{glb}\{1 - \nu^{\sigma'}_{\mathfrak{I}}(A) \mid \sigma' \text{ an } x\text{-alt. to } \sigma\} = 1 + \mathrm{glb}\{-\nu^{\sigma'}_{\mathfrak{I}}(A) \mid \sigma' \text{ an } x\text{-alt. to } \sigma\}$
4. $1 + \mathrm{glb}\{-\nu^{\sigma'}_{\mathfrak{I}}(A) \mid \sigma' \text{ an } x\text{-alt. to } \sigma\} = 1 - \mathrm{lub}\{\nu^{\sigma'}_{\mathfrak{I}}(A) \mid \sigma' \text{ an } x\text{-alt. to } \sigma\}$
5. $1 - \mathrm{lub}\{\nu^{\sigma'}_{\mathfrak{I}}(A) \mid \sigma' \text{ an } x\text{-alt. to } \sigma\} = 1 - \nu^{\sigma}_{\mathfrak{I}}(\exists x A)$
6. $1 - \nu^{\sigma}_{\mathfrak{I}}(\exists x A) = \nu^{\sigma}_{\mathfrak{I}}(\neg \exists x A)$

Let $X = \{1 - \nu^{\sigma'}_{\mathfrak{I}}(A) \mid \sigma' \text{ an } x\text{-alt. to } \sigma\}$, let $Y = \{-\nu^{\sigma'}_{\mathfrak{I}}(A) \mid \sigma' \text{ an } x\text{-alt. to } \sigma\}$. For step (3) we prove that $\mathrm{glb}X = 1 + \mathrm{glb}Y$. If $y \in Y$, then $\mathrm{glb}Y \leq y$ by def. so $1 + \mathrm{glb}Y \leq 1 + y$ and since $X = \{1 + y \mid y \in Y\}$ this makes $1 + \mathrm{glb}Y$ a lower bound of X. For any z which is a lower bound of X we have $z \leq 1 + y$ for all $y \in Y$ by def. of Y and so $z - 1 \leq y$ for all $y \in Y$ and so $z - 1 \leq \mathrm{glb}Y$, hence $z \leq 1 + \mathrm{glb}Y$.

Let Y be as above and let $Z = \{\nu^{\sigma'}_{\mathfrak{I}}(A) \mid \sigma' \text{ an } x\text{-alt. to } \sigma\}$. For step (4) we prove that $\mathrm{glb}Y = -\mathrm{lub}Z$. If $z \in Z$, then $z \leq \mathrm{lub}Z$ by def. so $-\mathrm{lub}Z \leq -z$ and since $Y = \{-z \mid z \in Z\}$ that makes $-\mathrm{lub}Z$ a lower bound of Y. For any x which is a lower bound of Y we have $x \leq -z$ for all $z \in Z$ by def. of Y and so $z \leq -x$ for all $z \in Z$ and so $\mathrm{lub}Z \leq -x$, hence $x \leq -\mathrm{lub}Z$.

Fact: $\exists x \neg A \vDash_{\aleph} \neg \forall x A$ and $\neg \forall x A \vDash_{\aleph} \exists x \neg A$

Proof: An exercise for the reader, but the argument is similar to the last proof.

Fact: $\exists x Px, \forall x(Px \rightarrow Qx) \vDash_{\aleph} \exists x Qx$

Proof: Fix $\mathfrak{I} = \langle D, \nu \rangle$ and σ such that $\nu^{g}_{\mathfrak{I}}(\exists x Px) = 1$ and $\nu^{g}_{\mathfrak{I}}(\forall x(Px \rightarrow Qx)) = 1$. By semantics of $\exists$ we have $\text{lub}\{\nu^{g'}_{\mathfrak{I}}(Px) \,|\, \sigma'$ an x-alt. to $\sigma\} = 1$ which means that $\nu^{g'}_{\mathfrak{I}}(Px) = 1$ on every x-alt. assignment to σ, and since any assignment is an x-alt. to itself we have $\nu^{g}_{\mathfrak{I}}(Px) = 1$. By the semantics of $\forall$ we have that $\text{glb}\{\nu^{g'}_{\mathfrak{I}}(Px \rightarrow Qx) \,|\, \sigma'$ an x-alt. to $\sigma\} = 1$ which means that $\nu^{g'}_{\mathfrak{I}}(Px \rightarrow Qx) = 1$ on every x-alt. assignment to σ, hence $\nu^{g'}_{\mathfrak{I}}(Px) \leq \nu^{g'}_{\mathfrak{I}}(Qx)$ on every such assignment, and since any assignment is an x-alt. to itself we have $\nu^{g}_{\mathfrak{I}}(Px) \leq \nu^{g}_{\mathfrak{I}}(Qx)$. Thus $\nu^{g}_{\mathfrak{I}}(Qx) = 1$ and so $\text{lub}\{\nu^{g'}_{\mathfrak{I}}(Qx) \,|\, \sigma'$ and a-alt. to $\sigma\} = \nu^{g}_{\mathfrak{I}}(\exists x Qx) = 1$.

Fact: $\nvDash_{\aleph} (\exists x Px \wedge \forall x(Px \rightarrow Qx)) \rightarrow \exists x Qx$

Proof: For a countermodel, fix an interpretation $\mathfrak{I} = \langle D, \nu \rangle$ with the two object domain $D = \{o_1, o_2\}$ interpreting predicates as $\nu(P)(o_1) = 0.9$ and $\nu(P)(o_2) = 0.6$ and $\nu(Q)(o_1) = 0.6$ and $\nu(Q)(o_2) = 0.3$. Then given arbitrary assignment σ we have $\nu^{g}_{\mathfrak{I}}(\exists x Px) = \text{lub}\{\nu^{g'}_{\mathfrak{I}}(Px) \,|\, \sigma'$ an x-alt. to $\sigma\} = \text{lub}\{0.9, 0.6\} = 0.9$ and $\nu^{g}_{\mathfrak{I}}(\forall x(Px \rightarrow Qx)) = \text{glb}\{\nu^{g'}_{\mathfrak{I}}(Px \rightarrow Qx) \,|\, \sigma'$ and x-alt. to $\sigma\} = \text{glb}\{0.7\} = 0.7$, but $\nu^{g}_{\mathfrak{I}}(\exists x Qx) = \text{lub}\{\nu^{g'}_{\mathfrak{I}}(Qx) \,|\, \sigma'$ an x-alt. to $\sigma\} = \text{lub}\{0.6, 0.3\} = 0.6$, so it follows thats $\nu^{g}_{\mathfrak{I}}((\exists x Px \wedge \forall x(Px \rightarrow Qx)) \rightarrow \exists x Qx) = 1 - (0.7 - 0.6) = 0.9$.

As I said, most of the unusual behavior here is due to the propositional fragment. The last two facts are just instances of the result we noted last week that the rule form of modus ponens for the Łukasiewicz arrow holds while the conditional form does not. Finally, I'll note the following standard rule for identity.

Fact: $a = b,\ A_{[a/x]} \vDash_{\aleph} A_{[b/x]}$

Discussion: This holds for fairly obvious reasons given how we have forced identity to effectively behave in the familiar classical way, but we could define a similar logic in which the identity predicate is a 'fuzzy' predicate like all of the non-logical predicates. This fuzzy identity would not satisfy the above inference. How to appropriately define fuzzy identity and how to philosophically understand the notion of fuzzy identity are left as exercises.