Ł$_{\aleph}$ Fuzzy Logic, Part I

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Modus Ponens and Vagueness

Where $\supset$ is the familiar material conditional, the following form of inference called material *modus ponens* is valid in classical logic.

$$\begin{array}{l} A \supset B \\ A \\ \hline B \end{array}$$

Vague predicates tolerate small differences, e.g. the predicate 'is red' can be true of two objects that vary slightly in hue. It seems that there is no sharp cutoff between the red things and the non-red things. Any small, discrete change is insufficient to change the color from red to non-red. If this is all correct, it would seem to pose a challenge to classical logic. The standard example of this challenge is called the Sorites series from the greek *soros* for 'is a heap'.

Take a single grain of sand. It alone is not a heap. The predicate 'is a heap' is vague, there is no sharp cutoff between heaps and non-heaps. In particular, a single grain of sand cannot make the difference, so every conditional of the form "if n grains of sand is not a heap, then $n+1$ grains of sand is not a heap" is true. But then by applications of modus ponens it follows that 6 million grains of sand are not a heap. And that is clearly false. We have an inference in which all of the premises seem to be true, the conclusion seems to be false, yet classical logic tells us the inference is valid.

Several responses to the Sorites are reasonable. The logic we are going to look at today, a system defined by Jan Łukasiewicz, can be understood as affording one type of response to the Sorites problem. In the semantics for this logic Ł$_{\aleph}$ we replace the Boolean semantic values $\{0, 1\}$ with a continuum of semantic values $[0, 1]$. The intuitive idea behind this is that the values 1 and 0 represent truth and falsity as usual, whereas the values in between represent degrees of 'indeterminacy' between truth and falsity. This makes the material conditional behave in undesirable ways, hence we supplement the language with a 'stronger' conditional which will be written with the arrow $\rightarrow$.

Propositional $Ł_{\aleph}$ Syntax and Semantics[1]

We work in a formal language the syntax of which consists of...

1. propositions p, q, r
2. connectives $\neg$, $\vee$, $\wedge$, $\rightarrow$
3. brackets $(\ )$

The formation rules for sentences are...

1. All propositions are (atomic) sentences.
2. If A is a sentence, then $\neg A$ is a sentence.
3. If A and B are sentences, so too are $A \wedge B$, $A \vee B$, $A \rightarrow B$.
4. Nothing is a sentence except by (1)-(3)

A propositional $Ł_{\aleph}$ valuation ν assigns values to sentences as follows...

1. $\nu(A) \in [0, 1]$ for atomic sentence A.
2. $\nu(\neg A) = 1 - \nu(A)$.
3. $\nu(A \wedge B) = \min\{\nu(A), \nu(B)\}$.
4. $\nu(A \vee B) = \max\{\nu(A), \nu(B)\}$.
5. $\nu(A \rightarrow B) = \begin{cases} 1 \text{ if } \nu(A) \leq \nu(B) \\ 1 - (\nu(A) - \nu(B)) \text{ if } \nu(A) > \nu(B) \end{cases}$

[1] Originally given in Łukasiewicz & Tarski, “Untersuchungen über den Aussagenkalkül", *Comptes rendus des séances de la Société des Sciences et des Lettres de Varsovie*, 23 (1930).

The material conditional and biconditional can be treated as defined connectives in the usual way, e.g. in every Ł$_\aleph$ valuation ν we have that $\nu(A \supset B) = \nu(\neg A \vee B)$ and $\nu(A \equiv B) = \nu((A \supset B) \wedge (B \supset A))$. We can also introduce a strong biconditional defined as $\nu(A \leftrightarrow B) = ((A \rightarrow B) \wedge (B \rightarrow A))$.

We define sentence satisfaction in the usual way. The sentence A is satisfied under Ł$_\aleph$ valuation ν iff $\nu(A) = 1$. We then say that a set of sentences Σ is satisfied under Ł$_\aleph$ valuation ν iff every sentence in Σ is satisfied under ν.

Semantic definition of propositional Ł$_\aleph$ consequence ($\models_\aleph$).

$\Sigma \models_\aleph A$ iff in every valuation under which Σ is satisfied, A is satisfied.

As in the familiar semantics for classical logic, only the true sentences are satisfied. What the continuum of degrees of indeterminate truth give us are an infinite number of ways to be unsatisfied other than being determinately false. One notable result is that the principle of Excluded Middle is invalid.

Fact: $\not\models_\aleph \neg A \vee A$

Proof: Fix valuation ν such that $\nu(A) = 0.5$, then $\nu(\neg A) = 1 - \nu(A) = 0.5$ and so $\nu(\neg A \vee A) = \max\{\nu(\neg A), \nu(A)\} = 0.5$. Thus, it is not the case that every sentence of the form $\neg A \vee A$ is satisfied under every valuation.

Note also that on the going definition of the material conditional, in any valuation ν we have that $\nu(A \supset A) = \nu(\neg A \vee A)$ and so we get the immediate and surprising result that so-called Material Identity is invalid.

Fact: $\not\models_\aleph A \supset A$

Clearly any sentence is a logical equivalent of itself, but since Material Identity fails we cannot hope to use the material biconditional $\equiv$ to express this logical equivalence. We can look at this as one natural motive for introducing the stronger conditional $\rightarrow$ into the language since we will have...

Fact: $\models_\aleph A \rightarrow A$

Proof: Fix any valuation ν, then the value of $\nu(A) \leq \nu(A)$ because $\nu(A) = \nu(A)$ and so by the given semantics for the strong Ł$_\aleph$ conditional, $\nu(A \rightarrow A) = 1$.

Some Notable Inferences

Fact: $A,\ A \to B \vDash_{\aleph}\ B$

Proof: Fix a $Ł_{\aleph}$ valuation ν such that $\nu(A) = 1$ and $\nu(A \to B) = 1$. By the semantics for $\to$, if $\nu(A \to B) = 1$, then we have $\nu(A) \leq \nu(B)$ and so $\nu(B) = 1$.

Fact: $A \to B,\ B \to C \vDash_{\aleph}\ A \to C$

Proof: Fix a $Ł_{\aleph}$ valuation ν such that $\nu(A \to B) = 1$ and $\nu(B \to C) = 1$. Then $\nu(A) \leq \nu(B)$ and $\nu(B) \leq \nu(C)$ and so $\nu(A) \leq \nu(C)$ which implies $\nu(A \to C) = 1$.

Fact: $A \to B \vDash_{\aleph}\ (A \wedge C) \to B$

Proof: Fix a $Ł_{\aleph}$ valuation ν such that $\nu(A \to B) = 1$, then $\nu(A) \leq \nu(B)$. By the semantics for conjunction $\nu(A \wedge C) = \min\{\nu(A), \nu(C)\}$ and so $\nu(A \wedge C) \leq \nu(B)$. Thus, $\nu((A \wedge C) \to B) = 1$.

Fact: $A \to \neg A \nvDash_{\aleph}\ \neg A$

Proof: For a countermodel, just fix a $Ł_{\aleph}$ valuation ν such that $\nu(A) = 0.4$. Then we have $\nu(\neg A) = 0.6$ and so $\nu(A \to \neg A) = 1$. This is a valuation in which the premise of the inference is satisfied, but the conclusion is not.

Fact: $\nvDash_{\aleph}\ (A \wedge (A \to B)) \to B$

Proof: For a countermodel, just fix a $Ł_{\aleph}$ valuation ν such that $\nu(A) = 0.6$ and $\nu(B) = 0.4$. Then we have $\nu(A \to B) = 0.8$ and so we have $\nu(A \wedge (A \to B)) = 0.6$ and thus $\nu((A \wedge (A \to B)) \to B) = 0.8$ which is the desired counterexample.

Fact: $\nvDash_{\aleph}\ (A \to (A \to B)) \to (A \to B)$

Proof: For a countermodel, just fix a $Ł_{\aleph}$ valuation ν such that $\nu(A) = 0.6$ and $\nu(B) = 0.4$. Then we have $\nu(A \to B) = 0.8$ and so $\nu(A \to (A \to B)) = 1$ and so $\nu((A \to (A \to B)) \to (A \to B)) = 0.8$ which is the desired counterexample.

Propositional Ł$_\aleph$ Proof Theory[2]

Unfortunately there is no tableaux system for Ł$_\aleph$ logic, but there is a complete axiomatization. It is not particularly easy to work with so I will just note it here. The system consists of the following six axioms and a single rule.

A1. $(A \to B) \to ((B \to C) \to (A \to C))$

A2. $A \to (B \to A)$

A3. $(A \to \neg B) \to (B \to \neg A)$

A4. $((A \to B) \to B) \to ((B \to A) \to A)$

A5. $((A \to B) \to B) \leftrightarrow (A \vee B)$

A6. $(A \wedge B) \leftrightarrow \neg(\neg A \vee \neg B)$

R1. From A and $A \to B$ infer B (Arrow Modus Ponens or AMP).

Proof theoretic definition of propositional Ł$_\aleph$ consequence ($\vdash_\aleph$).

> $\Sigma \vdash_\aleph A$ iff there is a finite list of Ł$_\aleph$ sentences ending in A such that each sentence in the list is either an axiom, a member of the set of sentences Σ, or follows from prior sentences in the list by AMP.

Responding to the Sorites

How does Ł$_\aleph$ help 'resolve' the Sorites? On possibility is to see the Sorites inference as valid but *unsound*. In particular, the indeterminacy of the transition from truth to falsity wrt vague claims means that while each conditional of the form "if n grains of sand is not a heap, then $n+1$ grains of sand is not a heap" seems true, it is not. We can model this by letting sentences h_i for each $i \in \mathbb{N}$ be "i grains of sand are a heap" and letting $\nu(h_i) = \frac{i}{1{,}000{,}000}$. Then the sentence $\nu(h_{1{,}000{,}000}) = 1$ as expected and the sentence $\nu(h_0) = 0$ as expected, but each conditional $\nu(h_n \to h_{n-1}) = 0.999\ldots$ making it almost, but not quite true.

[2]Various proofs of completeness were given by Rose & Rosser and C.C. Chang; several methods are surveyed in Rosser, "Axiomatization of Infinite Valued Logics", *Logique et Analyse* (1960).