Minutes: UConn Logic Group: December 5, 2008

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The session consisted of discussion (continued from the last meeting) of Greg Restall's (1994) paper: "Arithmetic and Truth in Łukasiewicz's Infinitely Valued Logic", as well as discussion of a (2000) paper, "The Liar Paradox and Fuzzy Logic", by Petr Hájek, Jeff Paris, and John Shepherson.

1.) Reed guided us through one final aspect of the Restall paper, namely how we get the formula R:

$R(x, y)$ formula: for any n, $p \in \mathbb{N}$
$\vdash R(\underline{n}, \underline{p}) \Leftrightarrow p = \ulcorner A_n \urcorner$
$\vdash \neg R(\underline{n}, \underline{p}) \Leftrightarrow p \neq \ulcorner A_n \urcorner$

Define a primitive recursive function: $h(u, v) = \mathbb{N}^2 \to \mathbb{N}$ by
$h(u, 0) = \ulcorner \neg \forall x \exists y (u^*(x+1, y) \wedge T(y)) \urcorner$ where u^* = formula φ s.t. $\ulcorner \varphi \urcorner = u$
$h(u, n+1) = \ulcorner h(u, 0)^* \circ h(u, n)^* \urcorner$
That is:
$h(u, 1) = \ulcorner h(u, 0)^* \circ h(u, 0)^* \urcorner$
$h(u, 2) = \ulcorner h(u, 0)^* \circ h(u, 1)^* \urcorner = \ulcorner h(u, 0)^* \circ h(u, 0)^* \circ h(u, 0)^* \urcorner$
...
$h(u, n) = \ulcorner h(u, 0)^* \circ h(u, 0)^* \circ ... \circ h(u, 0)^* \urcorner$, the $(n+1)$-fold fusion

Because $h(u, v)$ is primitive recursive, there is an arithmetic formula $H(u, v, w)$ s.t. for all $u, v, w \in \mathbb{N}$
$h(u, v) = w \Leftrightarrow \vdash H(u, v, w)$
$h(u, v) \neq w \Leftrightarrow \vdash \neg H(u, v, w)$ where provability is taken in the Peano arithmetic fragment (so crisp values are maintained in $Ł_\infty^\#$)

The fixed point lemma tells us there is a formula R s.t.
$\vdash Rvw \leftrightarrow H(\ulcorner R \urcorner, v, w)$

In sum, for $p, n \in \mathbb{N}$:

$$p = \ulcorner A_n \urcorner \Rightarrow h(\ulcorner R \urcorner, n) = p \Rightarrow \vdash H(\ulcorner R \urcorner, \underline{n}, \underline{p}) \Rightarrow \vdash R(\underline{n}, \underline{p})$$
$$p \neq \ulcorner A_n \urcorner \Rightarrow h(\ulcorner R \urcorner, n) \neq p \Rightarrow \vdash \neg H(\ulcorner R \urcorner, \underline{n}, \underline{p}) \Rightarrow \vdash \neg R(\underline{n}, \underline{p})$$

2.) Reed guided us through the results of the Hájek, Paris, and Shepherson paper:

Given the arithmetic axioms + the schema $\varphi \leftrightarrow Tr(\ulcorner\varphi\urcorner)$

Consider $\varphi_1, ...\varphi_n$ of axioms of form $\psi_1 \leftrightarrow Tr(\ulcorner\psi_1\urcorner), ...\psi_n \leftrightarrow Tr(\ulcorner\psi_n\urcorner)$. We build a standard model for the arithmetic axioms plus this finite list of truth axioms.

We can build a Model:

$\mathbb{N}$ + Interpretation Tr

Specify: $\nu(Tr(\underline{k}))$ for all k

$\nu(Tr(\ulcorner\alpha\urcorner)) = 0$ for all $\alpha \neq \psi_1, ...\psi_n$

$\nu(Tr(\ulcorner\psi_i\urcorner)) = x_i$ and we must make sure: $\nu(\psi_i) = x_i$

Model $= M(x_1, ...x_n)$, let $\bar{x} = x_1, ...x_n$, and set $\nu(Tr(\ulcorner\psi_i\urcorner)) = x_i$

$\nu_{M(\bar{x})}(\psi_i) : [0,1]^n \to [0,1]$ is continuous for all i

Let $f : [0,1]^n \to [0,1]^n$ be the continuous function

$f(x_1, ...x_n) = (\nu_{M(\bar{x})}(\psi_1), \nu_{M(\bar{x})}(\psi_2), ...\nu_{M(\bar{x})}(\psi_n))$

We want to find $r_1, r_2, ...r_n$ s.t. $f(r_1, r_2, ...r_n) = (r_1, r_2, ...r_n)$

Brouwer's Fixed Point Theorem we gives us that such $(r_1, r_2, ...r_n)$ exists.

Setting $\nu(\psi_i) = r_i$ gives a model for our finite set of truth axioms

Finally, if we have not only $\varphi \leftrightarrow Tr(\ulcorner\varphi\urcorner)$ but also:
$Tr(\ulcorner\varphi\urcorner) \leftrightarrow \neg Tr(\ulcorner\varphi\urcorner)$
$Tr(\ulcorner\varphi \circ \psi\urcorner) \leftrightarrow (Tr(\ulcorner\varphi\urcorner) \circ Tr(\ulcorner\psi\urcorner))$

Then, we get inconsistency by considering the formula λ defined in the paper to satisfy $\lambda \leftrightarrow \exists z(Tr(z \times \neg\lambda))$ where $z \times \neg\lambda$ is the z times fusion of $\neg\lambda$. The paper gives a derivation of both $\vdash \lambda$ and $\vdash \neg\lambda$.

Other notes:

- Closing discussion summary: we can consistently add the T-schema though not full intersubstitutability of $Tr(\ulcorner\alpha\urcorner)$ and α, but we get omega-inconsistency; on top of this, adding full intersubstitutability yields inconsistency

- In discusion, Lionel suggested that it would be interesting to look at Vann McGee's inconsistency result in "Truth, Vagueness, and Paradox" in connection with the results from Restall and Hájek, Paris, and Shepherson. (McGee's result shows that we can, with consistency and omega-consistency, add $Tr(\ulcorner\alpha\urcorner)$ for each true α in arithmetic, but we can't add the T-sentences without engendering inconsistency. Jc mentioned that while he hasn't looked at the result in a while and doesn't remember the details, you don't have full intersubstitutability since – via the validity of $\alpha \supset \alpha$ in classical arithmetic – you'd then get the T-biconditionals. His suspicion was that one salient place where the intersubstitutability broke down was w.r.t. negation and truth: $Tr(\ulcorner\neg\alpha\urcorner)$ and $\neg Tr(\ulcorner\alpha\urcorner)$ wouldn't be equivalent... But he'd want/need to check the details.

- It was suggested by Yael, and approved by all other members that the group's meetings during next semester will consist of presentations of work in progress by group members.